The Computation of Optimal Singular Bang-Bang Control I: Linear Systems

A general algorithm for the solution of both singular and bang-bang control problems is presented. The algorithm utilizes a limiting process and the solution of a constrained linear-quadratic control problem. The algorithm is applied to problems with both fixed and nonfixed final times. General numerical results for several linear system examples are presented, and two minimum time examples are discussed in detail.

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SCOPE

Control theorists have been aware of dynamic optimization problems with optimal singular and bang-bang solutions since the application of the calculus of variations to control problems. Singular solutions in optimal control have been thought by many people to be of only academic interest. However, Siebenthal and Aris (1964) have shown that optimal singular arcs occur in chemical reactor startup problems, hence there is some practical importance attached to this class of control problems. Other investigators, including Jackson (1966), Dyson and Horn (1967), and Ko and Stevens (1971), have also observed the occurrence of optimal singular arcs for different chemical reaction sequences and different mechanisms of control for chemical reactors.

Optimal bang-bang solutions, which can occur in conjunction with optimal singular arcs, have been considered within the framework of the maximum principle by Pontryagin (1962). He has shown that bang-bang control is optimal for the linear minimum time problem. Due to the existence of control constraints and control discontinuities, the computation of optimal bang-bang control presents a number of analytical and numerical difficulties. The singular case also requires extensive and careful study, since it appears as a degenerate form in the maximum principle formulation. The development of an effective computational algorithm to solve both bang-bang and singular control problems therefore presents a serious challenge.

The work of Weber and Lapidus (1971) with constrained dynamic systems surmounts the difficulties presented by constraints in the computation of an optimal control. Their computational algorithm utilizes the well-known solution of the linear-quadratic control problem. In this work the method of Weber and Lapidus is augmented and extended to solve singular/bang-bang control problems and in particular the minimum time problem. The proposed algorithm utilizes a limiting process originally suggested by Jacobson et al. (1969). A number of examples have been computed to test the reliability of the algorithm for solving large dimensional systems with multiple controls. This type of system complexity is characteristic of chemical engineering processes. Very few methods presently exist which can handle such problems effectively.

CONCLUSIONS AND SIGNIFICANCE

The present work demonstrates a new general algorithm for the solution of singular/bang-bang optimal control problems. The algorithm has been applied successfully to both linear and nonlinear systems and for fixed and nonfixed final times. Included in the general algorithm is a fast and reliable method for computing linear system minimum time control, which has been used to obtain the time optimal control of a sixth-order absorber with two controls and an eleventh-order nuclear reactor with one control. The algorithm utilizes a limiting process which obtains the singular/bang-bang solution as the limit of a series of nonsingular/nonbang-bang solutions. Discrete dynamic programming with penalty functions is used to solve this series of problems.

The general algorithm has the following advantages:

1. It readily solves control problems almost without regard for the dimension of the system equations, for example, an eleventh-order problem is solved in this work. It therefore does not have the limitations of phase plane analysis, as in the works of Siebenthal and Aris (1964), Jackson (1966), and Dyson and Horn (1967). Multiple controls are also readily handled. For linear systems minimum time problems, the computation is not restricted by the nature of the system eigenvalues. Chemical engineering processes are often characterized by large dimensionality and multiple inputs.

2. From all numerical experience, convergence of the algorithm is usually quite rapid, and although a good first guess will speed up the computation, the method requires no special first guess per se.

3. The algorithm utilizes the solution of the linear-quadratic control problem, the computation of which is based on a set of recursion formulas. The formulas are easy to implement on a digital computer and are part of a general purpose set of matrix programs at Princeton University [Automatic Synthesis Program, ASP, of Kal-

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man and Englar (1966)]. The ASP routine used in this study has been rewritten by Ellsworth (1969), and the storage requirements for the complete routine are not prohibitive (around 120k words of storage on the IBM 360/91.)

4. For linear systems, the algorithm is not subject to the slow rate of convergence exhibited by descent methods for singular/bang-bang control problems.

5. Since the nonfixed final time problem is solved as a series of fixed final time problems, there are many sets of meaningful and complete results for a single problem with the final time as a parameter. This approach is equivalent to flooding the final time parameter space.

The disadvantages of the general algorithm are:

1. For simple linear systems, the computation of minimum time control is not as fast as for other methods, for example, the method of Weischedel (1970).

2. Exact switching times are not calculated since the optimal controls are computed in discrete time. However, a formula for approximating the switching times is given in this work.

3. For nonlinear systems, the computation can be time-consuming; more details on nonlinear systems can be found in the sequel to this paper.

Singular/bang-bang optimal control occurs in control problems when the Hamiltonian is linear in the control. If the control variable is bounded, the optimal control from the optimality conditions of the maximum principle is bang-bang. When the Hamiltonian is not an explicit function of the control, that is, a variable multiplying the control in the Hamiltonian becomes zero for an interval of time, optimal singular control occurs. An explicit mathematical description of the singular/bang-bang case within the framework of the maximum principle can be found in Ko and Stevens (1971). Although the maximum principle gives the form of the optimal control, the control synthesis problem (explicit solution of the switching points, regions of optimal singular control) is still largely unsolved. Without the actual numerical solution of a given control problem, it is almost impossible to establish the existence of optimal singular arcs.

The new algorithm solves singular/bang-bang control problems via a limiting process. This limiting process was first suggested by Jacobson et al. (1969), and involves the augmenting of the performance index with an integral quadratic control functional multiplied by a variable coefficient $\left(\epsilon \int_{t_0}^{t_f} \mathbf{u}^T \mathbf{u} dt \right)$. With the addition of this term into the performance index, the problem loses its singular/bang-bang character. However, the variable coefficient ϵ is sequentially reduced until it approaches zero, and the original singular/bang-bang problem is solved in the limit. Therefore the singular/bang-bang solution is found as the limit of a series of nonsingular/nonbang-bang solutions.

An algorithm originally proposed by Weber and Lapidus (1971) is extended to solve the series of nonsingular/nonbang-bang problems. Their algorithm is applied at each stage of the limiting process, and hence an overlay structure of nonsingular/nonbang-bang problems exists for the original singular/bang-bang problem. Their method is based on converting the problem of interest into an augmented linear-quadratic problem (LQP), the solution of which is well-known and computationally feasible. The problem is solved in discrete time via discrete dynamic programming. Discrete penalty functions are used to handle two types of linear constraints—control and end point constraints. The penalty functions are quadratic, and thus the linear-quadratic form of the problem is preserved. However, the solution of the problem is now an iterative one due to the presence of the penalty functions, but convergence to the final solution is rapid. The solution of linear systems is optimal, but for nonlinear systems the resulting control is suboptimal. Therefore for nonlinear systems a descent method may be necessary to refine the suboptimal control to an optimal one.

The limiting process, when used in conjunction with Weber and Lapidus' method, is necessary in order to obtain the limiting bang-bang optimal or suboptimal controls; otherwise, convergence difficulties occur. When singular extremals are present, the limiting process actually speeds up the solution of the original problem, but its use is not necessary in order to obtain the solution. For nonlinear problems, the use of a second-order descent method to improve the suboptimal control requires the existence of a nonsingular H_{uu}. Adding a quadratic control term to the Hamiltonian guarantees a nonsingular H_{uu}, and the problem is made well-conditioned via the limiting process. First-order descent methods do not utilize the curvature H_{uu} and can be used without the limiting process, but these methods usually converge very slowly in the neighborhood of the optimum and require second order refinement.

If the final time for the control problem is unspecified, then the limiting process ($\epsilon \to 0$) and the solution of the LQP are imbedded within the optimization of the final time parameter. A method of solving minimum time problems, which is based on solving an LQP for a series of fixed final times, is demonstrated in this work. The iteration on the final time utilizes the unique properties of the Weber and Lapidus algorithm.

The structure of the general algorithm used to solve singular/bang-bang problems is as follows for the fixed final time and minimum final time cases.

Fixed Final Time: Linear State Equation and Quadratic Performance Index

Step 1: Discretize the state equation and performance index.

Step 2: Initiate the ϵ limiting process. Set ϵ^i $(i = 0)$, where i is an iteration counter. Augment the performance index with $\epsilon \sum_{k=1}^{N} \mathbf{u}^T{}_{k-1} \mathbf{R}_{k-1} \mathbf{u}_{k-1}$.

Step 3: For linear control constraints, define new state variables and augment the performance index with quadratic penalty functions. Specify control penalty function weighting coefficients ζ^m $(m = 0)$. For end point constraints, specify $\mathbf{Q}_N{}^m$ the penalty weighting matrix.

Step 4: Solve the augmented system (LQP) iteratively using discrete dynamic programming with penalty functions (DDP-PF).

Step 5: If the constraints are met to a specified tolerance, go to Step 6. Otherwise adjust penalty weighting coefficients ζ^m and $\mathbf{Q}_N{}^m$ $(m = m + 1)$. Return to Step 4.

Step 6: If $||u^0(\epsilon^i) - u^0(\epsilon^{i-1})|| > \delta$, then update $\epsilon^{i+1} = \alpha\epsilon^i, \alpha < 1$. Return to Step 3. If the convergence criterion is satisfied, convergence obtains, and the optimal singular/bang-bang control results.

Minimum Final Time: Linear State Equation

Step 1: Guess t_f^i $(i = 1, 2)$.

Step 2: Solve the fixed time constrained LQP by DDP-PF. Use the ϵ limiting process until ϵ convergence is obtained.

Step 3: If the computed optimal control is not bang-bang, generate t_f^{i+1} based on t_f^i and t_f^{i-1}. Return to Step 2. If the control is bang-bang, the minimum time control is obtained.

For the case of a nonlinear state equation and quadratic performance index with fixed or nonfixed final times, the structure of the calculations is very similar to that for a linear state equation. In fact, an iterative linearization of the state equation is performed, and the procedure given above for linear systems is imbedded in the computation of the optimal control for nonlinear systems. The part of the general algorithm devoted to nonlinear systems is described in the sequel to this paper.

The versatility of the general algorithm has been demonstrated by the solution of at least one example for each major case given above. The power of the algorithm is further magnified by the fact that the general algorithm readily handles almost any control problem, not just singular/bang-bang control problems. The original work of Weber and Lapidus along with the results obtained in this study show that the general algorithm has excellent potential.

THE DDP-PF ALGORITHM

For the particular case of linear control constraints

$$\boldsymbol{\alpha}^- \leq \mathbf{u}_k \leq \boldsymbol{\alpha}^+ \quad (k = 0,\ N-1) \tag{1}$$

and the state equation

$$\mathbf{x}_{k+1} = \boldsymbol{\Phi}_k \mathbf{x}_k + \boldsymbol{\Delta}_k \mathbf{u}_k \quad (k = 0,\ N-1) \tag{2}$$

one can imbed the constraints into the system by defining new state variables, thereby augmenting the original system. An equivalent unconstrained problem results through use of penalty functions for the constraint violations. The augmented performance index is

$$J_D = \sum_{k=1}^{N} [\mathbf{w}_k\, \boldsymbol{\Gamma}_k\, \mathbf{w}_k + \mathbf{u}^T{}_{k-1}\, \mathbf{R}_{k-1}\, \mathbf{u}_{k-1}] \tag{3}$$

Penalty weighting coefficients are contained in $\boldsymbol{\Gamma}_k$, and $\mathbf{w}_k$ is the augmented state vector, which consists of $\mathbf{x}_k$ and the extra state variables defined for the constraints. The work of Weber and Lapidus has demonstrated the construction of the augmented problem and its solution. The solution consists of iteratively solving an LQP via discrete dynamic programming, and for a linear state equation and linear control constraints, the resulting solution is optimal. For singular/bang-bang problems, the discrete time matrix $\mathbf{R}_{k-1}$ is sequentially reduced in the limiting process; $\mathbf{R}_{k-1}$ is the discrete time analog of ϵ.

The formation of the augmented system and performance index for linear control constraints as given above does not change significantly with the inclusion of linear end point constraints on the state variables. This type of constraint is encountered in the solution of minimum time problems. For the particular case of when the target state is the origin, that is, $\mathbf{x}(t_f) = \mathbf{0}$, one utilizes the final state matrix in the performance index as a penalty weighting matrix. Therefore one need only adjust the components of this matrix to force the components of the final state vector as close to zero as desired.

Although the penalty function method permits one to solve the constrained system as if it were unconstrained, the problem has an iterative nature. The iterative process is required for the following two reasons:

1. For each iteration, one must assume an interval of saturation (for both lower and upper control bounds), upon which the penalty functions are applied. The estimate of this interval of saturation must be updated from one iteration to the next. For singular/bang-bang control problems the successive approximation of the region of saturation is quite critical.

2. A sequence of penalty weighting coefficient matrices, which yield the desired degree of satisfaction of the constraints upon convergence, must be selected.

The computational steps utilized to solve the constrained LQP are the same as those used by Weber and Lapidus (1971) except in the termination of the computation. They end the penalty function calculations when the constraints are satisfied to some tolerance. The criterion which is used in this work is based on the region of saturation of the controls. When this region does not change from one iteration to the next, then the calculated controls for the two successive iterations are exactly the same. For the proper selection of the penalty weighting matrices, this is effectively the same as Weber's criterion for convergence.

A typical convergence pattern for a linear system bang-bang control problem with one switch, for the final stage of the limiting process, is depicted in Figure 1. Iteration 1 corresponds to the unconstrained control computation, and iteration 2 shows how the constraint violations from iteration 1 are penalized. However, iteration 2 results in new constraint violations, and these are penalized in iteration 3, etc. Iteration 5 has no new constraint violations, and the process converges at this point with no change in the

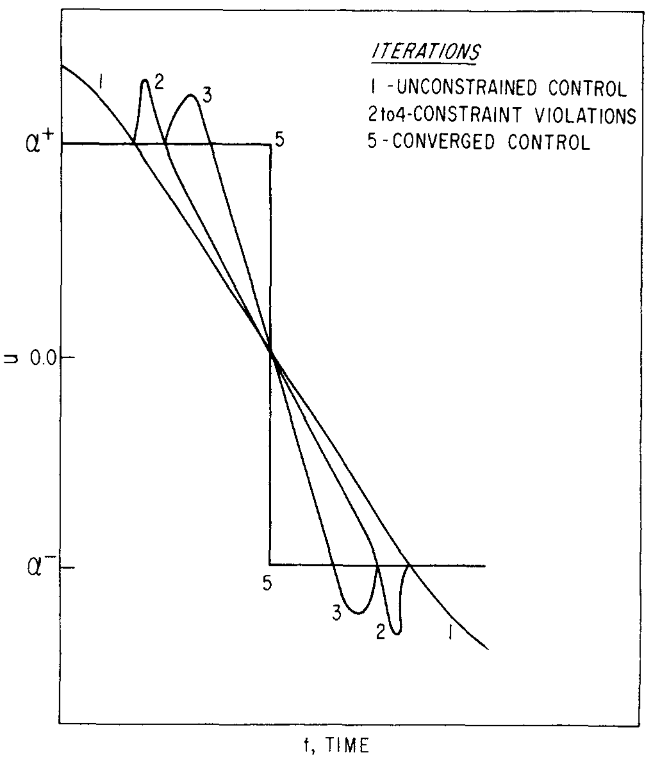

Fig. 1. Convergence of a linear system bang-bang control problem using the DDP-PF method.

region of control saturation. It should be noted that once a constraint violation is penalized, the corresponding control still violates the constraint, but only by a very small amount depending on the penalty weighting matrices. The controls in Figure 1 are represented as continuous curves for clarity, although the calculations are done in discrete time.

The DDP-PF method described above uses a totally different approach than other methods used in the literature to obtain optimal controls for linear systems. With this algorithm one satisfies the boundary conditions, the system equations, and the optimality equations obtained from discrete dynamic programming. The entity that is not satisfied is the control constraints. The iterative penalty function technique successively penalizes each nonfeasible control until all controls satisfy the constraints to a certain tolerance.

The preceding discussion of the DDP-PF algorithm applies to linear systems, in an optimal sense, and to nonlinear systems, in a suboptimal sense. Further discussion of nonlinear systems can be found in the second part of this work. The remainder of this first part is devoted to the optimal control of linear systems.

SOLUTION OF NON-FIXED FINAL TIME PROBLEMS WITH LINEAR STATE EQUATIONS

When the final time for a control problem is not explicitly specified and appears as a parameter in the performance index, the solution of the control problem becomes much more difficult. The minimum principle yields special requirements on the Hamiltonian for nonfixed final time problems. However, the optimal control synthesis is still very difficult and usually consists of an iteration on the final time. The particular type of nonfixed final time problem considered here is the minimum time problem.

The linear minimum time problem has received much attention in the literature. The methods of solution and representative studies have consisted of calculating switching functions (Oldenburger and Thompson, 1963), phase plane analysis (Douglas, 1966), search for initial adjoint conditions (Birta and Trushel, 1969), mathematical programming (Ho, 1962; Lesser and Lapidus, 1966), calculation of switching times (Yastreboff, 1969; Davison and Munro, 1970), and minimum amplitude control (Weischedel, 1970). Many of these approaches are limited to simple systems or real eigenvalued systems or exhibit numerical difficulties. Therefore it is desirable to develop an algorithm which can solve the minimum time problem with no restriction on system order or the system eigenvalues and which reduces numerical errors to a minimum.

It is demonstrated in this study that the minimum time problem can be solved as a sequence of fixed final time, constrained, LQP's. For each guessed final time, the DDP-PF algorithm is used to obtain the optimal control subject to control and end point constraints. The minimum time is the final time for which the calculated control is bang-bang, and the end point constraints are met to within a given tolerance. Therefore an overlay structure on the fixed final time problem exists for a minimum time problem. The restriction that the optimal control is bang-bang at the minimum time can be shown via the minimum principle; a short proof is given in Lapidus and Luus (1967). The fact that optimal singular control cannot occur for the linear minimum time problem is a direct consequence of the complete controllability of the system.

A new approach for iterating upon the final time is proposed here; it is realized through the unique properties of the DDP-PF algorithm. This approach uses the augmented system and performance index and is based upon setting the penalty weighting coefficients for the constraints at smaller values than those for the end point constraints. ϵ (or $\mathbf{R}_k$) in the quadratic control term is set at some small nonzero value, thus including some cost of control. These adjustments in the performance index for the augmented system yield the following results:

1. For final times larger than the minimum time, the control is unsaturated due to the inclusion of the quadratic control term, $\epsilon \int_0^{t_f} \mathbf{u}^T \mathbf{u}\, dt$. Both the end point and control constraints are satisfied for this case.
2. For final times smaller than the minimum time, the control becomes oversaturated because the end point constraints are satisfied to a greater degree than are the control constraints. This is due to the fact that the end point constraint violations are penalized more heavily; a tradeoff in the performance index occurs.

Representative values for the weighting coefficients are $\zeta_{i,k} = 1.0$ (control constraints) and $Q_{ii,N} = 10^2$ (end point constraints). ϵ can be set equal to 10^{-5}. It has been found empirically that these values can be established initially and do not need to be changed during the solution of the problem. Since the above initial value of ϵ will yield bang-bang control for the minimum time (that is, ϵ is close enough to zero), this procedure amounts to a one-step limiting process. This practice was implemented to save computation time.

One can define a degree of saturation, or control residual, R_c, as

$$R_c = \tau \sum_{k_1} (\alpha^+ - u_{k_1}) + \tau \sum_{k_2} (u_{k_2} - \alpha^-) \tag{4}$$

where τ = sampling interval
k_1 : those controls closest to the upper bound
k_2 : those controls closest to the lower bound

Therefore, for unsaturation R_c assumes a positive value, while for oversaturation R_c becomes negative. One merely needs to extrapolate or interpolate the control residuals using a regula falsi iteration until the degree of saturation becomes zero, which is equivalent to bang-bang control. At this point the minimum time is obtained. The iteration on the degree of saturation is unsophisticated, but it has proved to be very effective in obtaining the minimum time. More sophisticated interpolation techniques would probably speed up the calculation somewhat, but none have been investigated.

The power of this method for obtaining the minimum time is further magnified by the fact that it does not require a good first guess for the final time. Two guesses of the final time are required to start the method; the guesses can be quite far apart if desired. After utilizing the DDP-PF algorithm twice, the next final time is automatically calculated from the control residuals. The regula falsi iteration is characterized by rapid approach to the minimum time after only three guesses of the minimum time. If desired, good first guesses of the minimum time can be obtained by the algorithm of Gershwin (1969), who has obtained both upper and lower bounds on the minimum time via a Liapunov-like controllability theory for linear systems.

Numerical Results for Linear Systems

A number of numerical examples have been solved by the DDP-PF method for both fixed and nonfixed final times. The results for the fixed time problems, including simple systems also solved by Jacobson et al. (1969) and Pagurek and Woodside (1968) show that:

1. Except for the discrete representation of switches from one mode to another or from one control bound to another, the singular/bang-bang problem is accurately solved by the DDP-PF algorithm. In the vicinity of a switching point, an intermediate control results. For bang-bang control a good approximation of the switching time with an intermediate control at t_k is given by the following formula:

$$t_{sw} = t_k + |u(t_{k+1}) - u(t_k)| \cdot \tau/(\alpha^+ - \alpha^-) \quad (5)$$

This formula is basically a linear interpolation and is used to calculate switching times quoted in the numerical examples.

2. The limiting process ($\epsilon \to 0$) for problems with optimal singular arcs yields faster convergence to the optimum than that for no limiting process ($\epsilon = 0$); for strictly bang-bang extremals the limiting process is absolutely necessary in order to obtain convergence of the penalty function iterative process.

3. The DDP-PF method is not subject to the slow rate of convergence exhibited by descent methods for calculating singular/bang-bang control. The performance index is quite insensitive to the control in the neighborhood of the switching points, resulting in the slow convergence to the optimum. This difficulty was encountered by Pagurek and Woodside using the conjugate gradient method.

Complete information on the fixed time problem results can be found in Edgar (1971).

A special case of the fixed final time control problem is the infinite final time problem. Weber and Lapidus have solved an infinite time earth satellite problem which has a linear state equation and exhibits optimal bang-bang behavior. Weber and Lapidus successfully obtained the optimal control (with no limiting process) via the DDP-PF method; the control consists of distinct bang-bang and chattering modes. Thus this case will not be treated further in this work.

Minimum Time Examples

A number of linear minimum time examples are solved in this work. Two second-order systems, one with real eigenvalues and one with complex eigenvalues, have been studied, and the results show that the regula falsi iteration on the final time is an effective method of obtaining the minimum time. In both problems seven estimates of the final time yield a final time which is within 1% of the analytical minimum time. Also one can conclude that the proposed algorithm solves linear minimum time problems regardless of the nature of the system eigenvalues. Typical constraint violations are 10^{-5} for the controls (for $\zeta_{i,k} = 1.0$) and 10^{-5} for the final state variables (for $Q_{ii,N} = 10^2$). Further details on these two problems can be found in Edgar (1971).

The Linear Absorber. This example is chosen to test the algorithm's effectiveness in obtaining time optimal control of a system with two controls. The system is the six-plate linear absorber (sixth-order differential equation) as described in Lapidus and Luus (1967). Lesser and Lapidus (1966) have solved this problem as a sequence of fixed final time problems using the linear programming method to solve each subproblem. The work of Lesser and Lapidus was restricted by small machine storage capabilities (32k); under these storage conditions, Lesser's best estimate of the minimum time was $t_f = 6.0$.

The above work of Lesser and Lapidus permits a good first guess of the final time for the DDP-PF algorithm, and only two subsequent estimates of the final time are required until the minimum time of $t_{f\min} = 5.45$ is reached. The optimal control for this problem has one switch occurring for u_1 ($t_{sw} = 0.4796$) and four switches occurring for u_2 (1.649, 4.027, 4.588, and 5.340). At $t = 0.0$, $u_1 = 1.0$, and $u_2 = 0.972$, both numbers representing the upper constraints. Since the system eigenvalues are real, eigenvalue theory predicts at most five switches for each control ($n - 1$ switches, where n is the order of the system). It is important to note that a total of five switches occur for the two controls. The switching times listed above are calculated by Equation (5). The total computation time for solution is 24 seconds on the IBM 360/91 with an average of ten penalty function iterations required for each guess at the final time. All $|x_i\ (t_{f\min})|$ are less than 10^{-6} at the final time. Plots of the state trajectories of x_1, x_3, and x_6 for the minimum time control are given in Figure 2.

An interesting comparison is afforded by fixing u_1 at a steady state value and using only u_2 to control the absorber. For this particular case five switches in u_2 are observed, and the computed minimum time is 8.40. Therefore, the use of a second control reduces the minimum time significantly, while retaining the same total number of switches. The specification of a second control does not increase the computation time for the minimum time control substantially, and the incremental amount of computer storage for the two control problem is not significant.

The Nuclear Reactor. An eleventh-order model of a nuclear reactor with one control linearized about a particular operating point is the final minimum time example. This problem is selected as the ultimate test of state dimensionality for the proposed algorithm. This system was formulated by Cummins (1962) and has also been studied by Davison and Monro (1970). The complete control problem statement can be found in the latter work.

The system has one pair of complex eigenvalues, and hence the number of switches in the control is not known

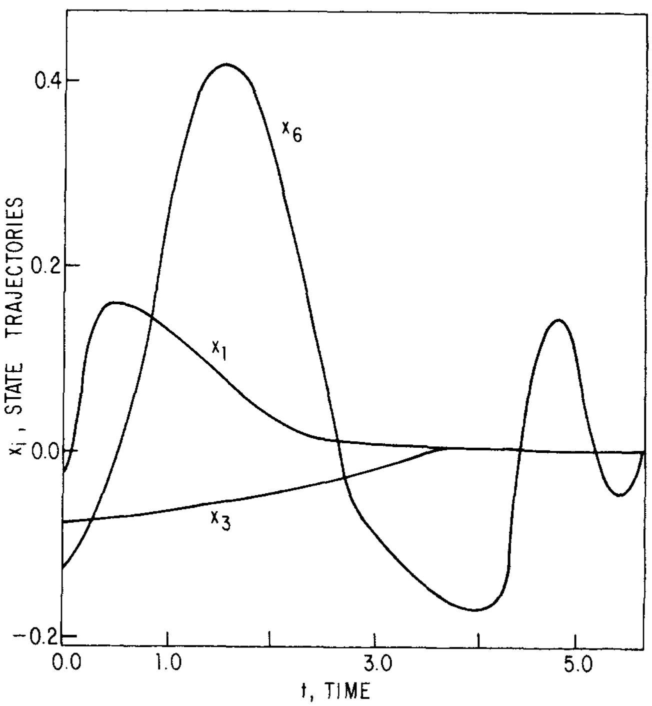

Fig. 2. Trajectories of x_1, x_3, and x_6 for the linear absorber $t_f = 5.45$; $\tau = 0.218$; $\zeta = 1.0$; $\epsilon = 10^{-6}$; $Q_N = 100I_6$.

a priori. The computed minimum time for this problem is $t_{f\min} = 2215$ with 80 time steps, and seven switches are observed for the time optimal control ($t_{sw} = 1449$, 1885, 1993, 2083, 2146, 2173, 2191). Seven estimates of the final time are required to obtain the minimum time. The control applied at $t = 0$ is $u = -1.0$. The last three switches are very close to each other, and this results in three successive intermediate controls. However, previous experience with intermediate controls allows the resolution of the switching times. The maximum component of $|x_1(t_{f\min})|$ is equal to 10^{-5}.

It is interesting to compare these results with those of Davison and Monro (1970), who have used a search for the explicit switching times, although the number of switches cannot be predicted for this problem. They have found ten switches and a minimum time of 2722.9 for this problem, but they have indicated that the resulting final states are not very close to the origin. They have postulated that more switches would be required to improve the answer, but the results from the general algorithm show that this hypothesis is incorrect. Davison's method apparently is unreliable when the number of switches is unknown. When the number of switches is large, the resulting multivariable optimization problem, that is, finding the optimal switching points, becomes very difficult using even the most effective search techniques.

The computation time required for the proposed algorithm to solve this example problem is 40 seconds (IBM 360/91). This amount of time is only double that required to solve the second-order system with real eigenvalues.

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NOTATION

$\mathbf{A}$ = matrix
$\mathbf{B}$ = matrix
d = parameter
H = Hamiltonian
$\mathbf{I}_n$ = $n \times n$ identity matrix
J = performance index
N = number of discrete stages of control
$\mathbf{Q}$ = matrix
$\mathbf{R}$ = matrix
R = control residual
s = parameter
$\mathbf{S}$ = matrix
t = time
$\mathbf{u}$ = control vector
$\mathbf{w}$ = augmented state vector
$\mathbf{x}$ = state vector

Greek Letters

α = control bound; α^+ = upper bound, α^- = lower bound
$\mathbf{\Delta}$ = matrix
ϵ = parameter in limiting process
$\mathbf{\Gamma}$ = matrix
$\mathbf{\Phi}$ = matrix
τ = discrete time sampling interval
ζ = control penalty weighting coefficient

Subscripts

C = continuous
D = discrete
f = final (time)
i, j = matrix or vector component
k = discrete time step index
min = minimum (time)
o = initial
sw = switch
$\mathbf{u}$ = partial derivative with respect to $\mathbf{u}$

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